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2001 J. Phys. A: Math. Gen. 34 5269

(http://iopscience.iop.org/0305-4470/34/25/309)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 5269–5281 PII: S0305-4470(01)14539-7

Scattering of scalar pulses on planes

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Received 8 June 2000, in final form 4 May 2001
Published 15 June 2001
Online at stacks.iop.org/JPhysA/34/5269

Abstract

The Fredholm integral equation approach (Hillion P 2001 *J. Phys. A: Math. Gen.* **34** 4687) of dealing with the scattering of scalar harmonic waves on plane obstacles is extended to two classes of scalar pulses with finite duration or decreasing strongly for $|t| \Rightarrow \infty$. The consistency of this formalism is checked for digital signals and focus wave modes impinging on a perfectly reflecting smooth plane. To illustrate this approach, two different topics are investigated: scattering of digital signals on an impedance plane and their reflection on a time-reversal mirror.

PACS numbers: 4120J, 0230N, 0230R, 0230U

1. Introduction

In a previous work [1], a Fredholm integral equation was developed to analyse the scattering by obstacles, mainly planes, of scalar harmonic waves; solutions of the Helmholtz equation and the differences from the conventional scattering integral equations were emphasized.

This approach is generalized here to the scattering by obstacles of scalar pulses, solutions of the wave equation, a problem that raised a little interest in the past [2–4] when technology generated pulses considered as packets of harmonic plane waves. However, the situation is now changing with the blossoming of digital signals, which leads us to consider two classes of scalar pulses: the $\mathcal{S}$-pulses of rapid descent such as for $|t| \Rightarrow \infty$, they and all their partial derivatives decrease strongly to zero at a given point in space and the $\mathcal{D}$-pulses, which are zero outside some finite interval of time. The corresponding Fredholm integral equations require two different types of Green function for $\mathcal{S}$- and $\mathcal{D}$-pulses.

We present in section 2 the time-dependent Fredholm integral equations for scattering of pulses on a perfectly reflecting smooth plane, then the differences with the conventional approach are stressed and the consistency of this approach is checked on two simple examples. To illustrate this approach, two different topics are discussed in section 3: scattering of $\mathcal{D}$-pulses on an impedance plane and reflection on a time-reversal mirror. The reasons to be interested in pulse scattering on planes are given in section 4.

2. Time-dependent Fredholm integral equations

2.1. General theory

We use the same notations as in [1]: $\boldsymbol{x}(xy, z)$ and $\boldsymbol{x}'(x', y', z')$ denote respectively the action and point sources in the Green functions while the surface S is written Σ, Σ', for the action and source points respectively and we consider scalar pulses depending arbitrarily on time and impinging on a perfectly reflecting smooth plane located at $z = 0$. We still assume that the total field

$$\psi(\boldsymbol{x}, t) = \psi_{\mathrm{i}}(\boldsymbol{x}, t) + \psi_{\mathrm{s}}(\boldsymbol{x}, t) \tag{1}$$

in which ψ_{i}, ψ_{s} are the incident and scattered fields and the Green function $G(\boldsymbol{x}, t; \boldsymbol{x}', t')$ satisfies the Dirichlet and Neumann boundary conditions (soft and hard in acoustics)

$$[\psi(\boldsymbol{x}, t)]_{z=0} = 0 \qquad [G_D(\boldsymbol{x}, t; \boldsymbol{x}', t')]_{z=0} = 0 \tag{2a}$$

$$[\partial_z \psi(\boldsymbol{x}, t)]_{z=0} = 0 \qquad [\partial_z G_N(\boldsymbol{x}, t; \boldsymbol{x}', t')]_{z=0} = 0 \tag{2b}$$

with the Green functions $G_D G_N$, deduced from the free-space Green function $G(\boldsymbol{x}, t; \boldsymbol{x}', t')$ to be discussed below using the method of images $\{\boldsymbol{x} = (x, y, z), \xi = (x, y, -z)\}$

$$\begin{aligned} G_D(\boldsymbol{x}, t; \boldsymbol{x}', t') &= G(\boldsymbol{x}, t; \boldsymbol{x}', t') - G(\xi, t; \boldsymbol{x}', t') \\ G_N(\boldsymbol{x}, t; \boldsymbol{x}', t') &= G(\boldsymbol{x}, t; \boldsymbol{x}', t') + G(\xi, t; \boldsymbol{x}', t'). \end{aligned} \tag{3}$$

Moreover, the Fredholm integral equations take the form

$$\psi(\boldsymbol{x}, t) = \int_{-\infty}^{\infty} \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}x' \int_{-\infty}^{\infty} \mathrm{d}y' \, [G_D(\boldsymbol{x}, t; \boldsymbol{x}', t') \partial_{z'} \psi(\boldsymbol{x}', t')]_{z'=0} \tag{4a}$$

$$\psi(\boldsymbol{x}, t) = -\int_{-\infty}^{\infty} \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}x' \int_{-\infty}^{\infty} \mathrm{d}y' \, [\psi(\boldsymbol{x}', t') \partial_{z'} G_N(\boldsymbol{x}, t; \boldsymbol{x}', t')]_{z'=0}. \tag{4b}$$

Now let $G(\boldsymbol{x}, \boldsymbol{x}'; k)$ denote the free-space Green function of the Helmholtz equation (equation (8*a*) in [1])

$$\begin{aligned} &G(\boldsymbol{x}, \boldsymbol{x}'; k) = (\mathrm{i}/8\pi^2) \iint_{-\infty}^{\infty} \mathrm{d}ß \, \mathrm{d}\gamma \, k_z^{-1} \exp[\mathrm{i}ß(x - x') + \mathrm{i}\gamma(y - y') + \mathrm{i}k_z|z - z'|] \\ &k_z^2 = k^2 - ß^2 - \gamma^2 \end{aligned} \tag{5}$$

then, for $\mathcal{S}$-pulses, $G(\boldsymbol{x}, t; \boldsymbol{x}', t')$ is the inverse Fourier transform of $G(\boldsymbol{x}, \boldsymbol{x}'; k)$

$$G(\boldsymbol{x}, t; \boldsymbol{x}', t') = (1/2\pi) \int_{-\infty}^{\infty} \exp[\mathrm{i}k(ct - ct')] G(\boldsymbol{x}, \boldsymbol{x}'; k) \, \mathrm{d}k \tag{6}$$

while for $\mathcal{D}$-pulses $G(\boldsymbol{x}, t; \boldsymbol{x}', t')$ is the inverse Laplace transform of $G(\boldsymbol{x}, \boldsymbol{x}'; k)$ with respect to the variable $s = \mathrm{i}k$

$$G(\boldsymbol{x}, t; \boldsymbol{x}', t') = (1/2\pi\mathrm{i}) \int_{\mathrm{Br}} \exp[s(ct - ct')] G(\boldsymbol{x}, \boldsymbol{x}'; s) \, \mathrm{d}s \tag{7}$$

in which the Bromwich contour Br is made up of a line L parallel to the imaginary axis of the complex s-plane with all the singularities of the integrand on its left.

Let us now make clear the differences between the present approach and the conventional one, in which to avoid confusion we denote by g the Green functions. The total field ψ still satisfies the boundary conditions (2*a*) and (2*b*) while the corresponding boundary conditions for $g_{D,N}$ are now given on the Σ'-plane $z' = 0$ and no longer on Σ

$$[g_D(\boldsymbol{x}, t, \boldsymbol{x}', t')]_{\Sigma'} = 0 \qquad [\partial_z g_N(\boldsymbol{x}, t; \boldsymbol{x}', t')]_{\Sigma'} = 0 \tag{8}$$

supplying the two integral equations

$$\psi(\boldsymbol{x},t)=\psi_{\rm i}(\boldsymbol{x},t)+\iiint_{-\infty}^{\infty}{\rm d}t'\,{\rm d}x'\,{\rm d}y'\,[g_N(\boldsymbol{x},t;\boldsymbol{x}',t')\partial_{z'}\psi_{\rm s}(\boldsymbol{x}',t')]_{z'=0} \tag{9}$$

$$\psi(\boldsymbol{x},t)=\psi_{\rm i}(\boldsymbol{x},t)-\iiint_{-\infty}^{\infty}{\rm d}t'\,{\rm d}x'\,{\rm d}y'\,[\psi_{\rm s}(\boldsymbol{x}',t')\partial_{z'}g_D(\boldsymbol{x},t;\boldsymbol{x}',t')]_{z'=0} \tag{10}$$

in which $g_{D,N}$ are obtained from the free-space Green function g, still by the method of images but now taken with respect to the Σ'-plane $z'=0$, so that with $\xi'(x',y';-z')$ we obtain

$$\begin{aligned}g_D(\boldsymbol{x},t,\boldsymbol{x}',t')&=g(\boldsymbol{x},t;\boldsymbol{x}',t')-g(\boldsymbol{x},t;\xi',t')\\ g_D(\boldsymbol{x},t;\boldsymbol{x}',t')&=g(\boldsymbol{x},t;\boldsymbol{x}',t')+g(\boldsymbol{x},t;\xi',t')\end{aligned} \tag{11}$$

where the Green function g [2,4] in which δ is the Dirac distribution is

$$g(\boldsymbol{x},t;\boldsymbol{x}',t')=(1/4\pi)\delta[c^{-1}|\boldsymbol{x}-\boldsymbol{x}'|-(t-t')]/|\boldsymbol{x}-\boldsymbol{x}'|. \tag{12}$$

It is assumed in (9) and (10) that $\psi_{\rm i}(\boldsymbol{x})$ or $\partial_z\psi_{\rm i}(\boldsymbol{x})$ is known on the plane $z=0$ so that from the boundary conditions (2*a*) and (2*b*) the expressions of $\psi_{\rm s}(\boldsymbol{x}')$ to be introduced in the integrand of (9) and (10) are also known. So, strictly speaking, as stressed in [1], equations (10*a*) and (10*b*) are not integral equations but they are solutions of the wave equation in an integral form.

2.2. *Consistency of Fredhom integral equations*

To check the consistency of Fredholm integral equations we investigate the reflection of $\mathcal{S}$- and $\mathcal{D}$-pulses on a plane mirror, taking a focus wave mode [5,6] as a prototype of $\mathcal{S}$-pulse, and for the sake of simplicity we consider a two-dimensional problem, assuming that the fields do not depend on y.

A two-dimensional scalar focus wave mode impinging on the plane $z=0$ from the region $z<0$ has the form [7,8] with $\boldsymbol{u}=(x,z)$

$$\psi_{\rm i}(\boldsymbol{u},t)=h^{-1/2}(\boldsymbol{u},t)\exp[{\rm i}\omega c^{-1}(ct-Z)-\omega X^2c^{-1}h^{-1}(\boldsymbol{u},t)] \tag{13}$$

$$h(\boldsymbol{u},t)=a-{\rm i}(ct+Z)\qquad Z=x\sin\theta+z\cos\theta\qquad X=x\cos\theta-z\sin\theta \tag{13a}$$

where the parameter a is an arbitrary positive length. So $\psi_{\rm i}$ is a $\mathcal{S}$-pulse and it is a simple exercise to check that $\psi_{\rm i}$ is solution of the two-dimensional wave equation. We assume that the total field satisfies the boundary condition (2*b*).

An interesting question is whether reflection at a perfectly conducting plane preserves the structure of focus wave modes so that reflected waves are obtained in agreement with the Descartes–Snell law by changing θ into $\pi-\theta$ (or equivalently z into $-z$) in equations (13) and (13*a*). If so, $\psi=\psi_{\rm i}+\psi_{\rm r}$ is solution of the integral equation (4*b*) that we now write since we are working in a two-dimensional space

$$\psi(\boldsymbol{u},t)=-\int_{-\infty}^{\infty}{\rm d}t'\int_{-\infty}^{\infty}{\rm d}x'\,[\psi(\boldsymbol{u}',t')\partial_{z'}G_N(\boldsymbol{u},t,\boldsymbol{u}',t')]_{z'=0}\qquad z\leqslant 0 \tag{14}$$

while substituting (5) into (6) gives with $k_z^2=k^2-ß^2$

$$4\pi^2G(\boldsymbol{u},t;\boldsymbol{u}',t')={\rm i}c\int_{-\infty}^{\infty}{\rm d}t'\int_{-\infty}^{\infty}{\rm d}ß\,k_z^{-1}\exp[{\rm i}k(ct-ct')+{\rm i}ß(x-x')+{\rm i}k_z|z-z'|] \tag{15}$$

so that substituting (15) into (3) and using the relation (equation (23) in [1])

$$2\pi[\partial_{z'}G_N(\boldsymbol{u},\boldsymbol{u}';k)]_{z'=0}=-\int_{-\infty}^{\infty}{\rm d}ß\exp[{\rm i}ß(x-x')]\cos(k_zz) \tag{15a}$$

we obtain

$$4\pi^2[\partial_{z'}G_N(\boldsymbol{u},t;\boldsymbol{u}',t')]_{z'=0} = -c\int_{-\infty}^{\infty}\mathrm{d}k\int_{-\infty}^{\infty}\mathrm{d}ß\,\exp[\mathrm{i}k(ct-ct')+\mathrm{i}ß(x-x')]\cos(k_z z). \tag{16}$$

Now according to (13) and (13a) we have $\psi_{\mathrm{i}}(\boldsymbol{u}',t')]_{z'=0} = [\psi_{\mathrm{r}}(\boldsymbol{u}',t')]_{z'=0}$ and

$$[\psi(\boldsymbol{u}',t')]_{z'=0} = 2(a-\mathrm{i}r')^{-1/2}\exp[\mathrm{i}\omega(t'-x'\sin\theta/c)-\omega c^{-1}x'^2\cos^2\theta(a-\mathrm{i}r')^{-1}] \tag{17}$$

$$r' = ct' + x'\sin\theta. \tag{17a}$$

Substituting (16) and (17) into (14) gives

$$2\pi^2\psi(\boldsymbol{u},t) = \iint_{-\infty}^{\infty}\mathrm{d}k\,\mathrm{d}ß\,\exp(\mathrm{i}kct+\mathrm{i}ßx)\cos(k_z z)F(ß,k) \tag{18}$$

$$F(ß,k) = \iint_{-\infty}^{\infty}\mathrm{d}t'\,\mathrm{d}x'(a-\mathrm{i}r')^{-1/2} \times\exp[\mathrm{i}ct'(\omega/c-k)-\mathrm{i}x'(ß+\omega\sin\theta/c)-c^{-1}\omega x'^2\cos\theta/(a-\mathrm{i}r')] \tag{19}$$

and one checks in appendix A that

$$\psi(\boldsymbol{u},t) = \psi_{\mathrm{i}}(\boldsymbol{u},t)+\psi_{\mathrm{r}}(\boldsymbol{u},t) \tag{20}$$

reflection at a mirror preserves the structure of focus wave modes, making the Descartes–Snell law valid.

Still assuming that the total field satisfies the boundary condition (2b) we now consider a scalar $\mathcal{D}$-pulse incident from the region $z<0$ of space normally on the plane S

$$\psi_{\mathrm{i}}(\boldsymbol{u},t) = f(t-z/c)[U(t-z/c)-U(t-\tau-z/c)] \tag{21}$$

in which U is the unit step function and f an arbitrary function with partial derivatives while τ is the duration of the incident pulse. Any point of S is reached by ψ_{i} at the same time $t=0$ at which a reflected pulse $\psi_{\mathrm{r}}(\boldsymbol{u},t)$ is generated, so one is only interested in the total field for $t\geqslant 0$, and assuming *a priori* that the Descartes–Snell law is valid we may write

$$\psi_{\mathrm{i}}(\boldsymbol{u},t) = f(t-z/c)U(t)[U(t-z/c)-U(t-\tau-z/c)] \tag{22a}$$

$$\psi_{\mathrm{r}}(\boldsymbol{u},t) = f(t+z/c)U(t)[U(t+z/c)-U(t-\tau+z/c)] \tag{22b}$$

and if we are correct $\psi_{\mathrm{i}}+\psi_{\mathrm{r}}$ should be the solution of the integral equation (14) in which the Green function G_N is now defined by the relations (3), (5) and (7) so that

$$8\mathrm{i}\pi^2G_N(\boldsymbol{u},t;\boldsymbol{u}',t') = c\int_{\mathrm{Br}}\mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}ß\,s_z^{-1}\Phi(ß,s)\{\exp(-s_z|z-z'|)+\exp(-s_z|z+z'|)\} \tag{23}$$

$$\Phi(ß,s) = \exp[s(ct-ct')+\mathrm{i}ß(x-x')] \qquad s_z = (s^2+ß^2)^{1/2}. \tag{23a}$$

Then, the derivative of G_N is

$$8\mathrm{i}\pi^2\partial_{z'}G_N(\boldsymbol{u},t;\boldsymbol{u}',t') = c\int_{\mathrm{Br}}\mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}ß\,\Phi(ß,s) \times\{\exp(-s_z|z-z'|)\partial_{z'}|z-z'|+\exp(-s_z|z+z'|)\partial_{z'}|z+z'|\} \tag{24}$$

and using the relations (equation (22) in [1])

$$\begin{aligned} &|z-z'|_{z'=0} = -z \qquad [\partial_{z'}|z-z'|]_{z'=0} = 1 \qquad z<0 \\ &|z+z'|_{z'=0} = z \qquad [\partial_{z'}|z+z'|]_{z'=0} = 1 \qquad z>0 \end{aligned} \tag{25}$$

we obtain

$$4\mathrm{i}\pi^2[\partial_{z'} G_N(\boldsymbol{u}, t; \boldsymbol{u}', t')]_{z'=0} = -c \int_{\mathrm{Br}} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}ß\, \Phi(ß, s) \cosh(s_z z) \tag{26}$$

while according to (22*a*) and (22*b*), discarding the useless repetition of the step function $U(t')$,

$$[\psi(\boldsymbol{u}', t')]_{z'=0} = [\psi_\mathrm{i}(\boldsymbol{u}', t') + \psi_\mathrm{r}(\boldsymbol{u}', t')]_{z'=0} = 2f(t')[U(t') - U(t' - \tau)]. \tag{27}$$

Substituting (26) into (16) and taking into account (23*a*) gives

$$\psi(\boldsymbol{u}, t) = (1/4\mathrm{i}\pi^2) \int_{\mathrm{Br}} \mathrm{d}s \exp(sct) \int_{-\infty}^{\infty} \mathrm{d}ß \exp(\mathrm{i}ßx) \cosh(s_z z) F(ß, s) \tag{28}$$

$$F(ß, s) = \int_{-\infty}^{\infty} \mathrm{d}x' \exp(-\mathrm{i}ßx') \int_{-\infty}^{\infty} c\, \mathrm{d}t' \exp(-sct')[\psi(\boldsymbol{u}', t')]_{z'=0} \tag{28a}$$

and taking into account (27)

$$\begin{aligned} F(ß, s) &= \int_{-\infty}^{\infty} \mathrm{d}x' \exp(-\mathrm{i}ßx') \int_0^{\tau} \mathrm{d}t' f(t') \exp(-sct') \\ &= 4\pi\delta(ß) \int_0^{\tau} \mathrm{d}t' f(t') \exp(-sct'). \end{aligned} \tag{28b}$$

Substituting (28*b*) into (28) gives

$$\begin{aligned} \psi(\boldsymbol{u}, t) &= (c/\mathrm{i}\pi) \int_0^{\tau} \mathrm{d}t' f(t') \int_{\mathrm{Br}} \mathrm{d}s \exp(sct - sct') \int_{-\infty}^{\infty} \mathrm{d}ß\, \delta(\beta) \exp(\mathrm{i}ßx) \cosh(s_z z) \\ &= \int_0^{\tau} \mathrm{d}t' f(t')[B_+(z, t') + B_-(z, t')] \end{aligned} \tag{29}$$

with

$$B_\pm(z, t) = (c/2\mathrm{i}\pi) \int_{\mathrm{Br}} \mathrm{d}s \exp[sct - sc(t' \pm z/c)] \tag{29a}$$

and using the well known Laplace transform formula [9] $L^{-1}\{\exp(-as)\} = \delta(t-a)$ for $a > 0$, we obtain

$$B_\pm(z, t') = \delta[t - (t' \pm z/c)]U(t' \pm z/c). \tag{30}$$

Substituting (30) into (29) gives $\psi(\boldsymbol{u}, t) = \psi_\mathrm{i}(\boldsymbol{u}, t) + \psi_\mathrm{r}(\boldsymbol{u}, t)$ since

$$\int_0^{\tau} \mathrm{d}t' f(t')U(t' - \pm z/c)\delta(t - t' \pm z/c) = f(t \pm z/c)U(t)[U(t \pm z/c) - U(t - \tau \pm z/c)] \tag{31}$$

the unit step functions in the square bracket arising from the fact that the integral on t' is non-null only if $t \pm z/c$ is in the interval $(0, \tau)$.Therefore, the *a priori* assumption that reflection brings no distortion to a $\mathcal{D}$-pulse is checked for normal incidence: a result also valid at any incidence.

2.3. Numerical experiments

A numerical evaluation of the integral equation (14) when the function $f(z, t)$ in the pulse (21) is $f(t - z/c) = \exp[-k^2(t - z/c)^2]$ and with the Green function (23). In all calculations, the wavenumber k and the light velocity c are unity. Tables 1 and 2 give the pulse amplitude $\psi(z, t)$ at the times $t = z$, $z + 2$ and $z + 4$ for $z = 0$ and 10 and for three values of the pulse duration τ.

One obtains the same results for $z = 20$ in agreement with the fact that the amplitude depends on $t - z$ provided that $t - z < \tau$.

Table 1. $z = 0$.

t	z	$z+2$	$z+4$
$\tau = 3$	1	18.3×10^{-3}	0
$\tau = 5$	1	18.3×10^{-3}	1.12×10^{-9}
$\tau = 7$	1	$1; 3 \times 10^{-3}$	1.12×10^{-9}

Table 2. $z = 10$.

t	z	$z+2$	$z+4$
$\tau = 3$	0.5	9.15×10^{-3}	0
$\tau = 5$	0.5	9.15×10^{-3}	56.2×10^{-9}
$\tau = 7$	0.5	9.15×10^{-3}	56.2×10^{-9}

3. Application

3.1. $\mathcal{D}$-pulses on an imperfectly reflecting plane

To illustrate this integral equation approach, we consider what happens when the plane S is not perfectly conducting, assuming that the Neumann boundary conditions (2*b*) are replaced by the somewhat more general boundary conditions in which d is a positive constant and s_z the transverse component of the wavevector

$$[\{\partial_z + s_z d\}\psi(\boldsymbol{u}, t)]_{z=0} = 0 \qquad [\{\partial_z + s_z d\}G_Z(\boldsymbol{u}, t; \boldsymbol{u}', t')\}]_{z=0} = 0. \tag{32}$$

From the definition (3), (5) and (7) of G_N we obtain at once

$$G_Z(\boldsymbol{u}, t; \boldsymbol{u}', t') = G_N(\boldsymbol{u}, t; \boldsymbol{u}', t') \exp(-s_z d|z - z'|) \tag{33}$$

since taking into account the relation $[\partial_z G_N]_{z=0} = 0$ a simple calculation gives

$$[\partial_z G_Z]_{z=0} = -s_z d[G_Z]_{z=0}[\partial_z|z - z'|]_{z=0} \tag{33a}$$

which implies (33) since $[\partial_z|z - z'|]_{z=0} = 1$ for $z' < 0$.

Then, for the boundary conditions (32) the integral equation (14) becomes with G_Z instead of G_N

$$\psi(\boldsymbol{u}, t) = -\int_{-\infty}^{\infty} \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}x' \, [\psi(\boldsymbol{u}', t')\partial_{z'} G_Z(\boldsymbol{u}, t; \boldsymbol{u}', t')]_{z'=0} = z \leqslant 0. \tag{34}$$

However, now there is no reason why the reflected pulse should be supplied by the Descartes–Snell law, so we substitute $\psi_{\mathrm{i}} + \psi_{\mathrm{r}}$ for ψ in (34) to obtain the integral equation satisfied by the reflected pulse and we obtain

$$\psi_{\mathrm{r}}(\boldsymbol{u}, t) + \phi_{\mathrm{r}}(\boldsymbol{u}, t) = -[\psi_{\mathrm{i}}(\boldsymbol{u}, t) + \phi_{\mathrm{i}}(\boldsymbol{u}, t)] \qquad z \leqslant 0 \tag{35}$$

in which $\phi_{\mathrm{i,r}}$ are the integrals

$$\phi_{\mathrm{i,r}}(\boldsymbol{u}, t) = \int_{-\infty}^{\infty} \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}x' \, [\psi_{\mathrm{i,r}}(\boldsymbol{u}', t')\partial_{z'} G_Z(\boldsymbol{u}, t; \boldsymbol{u}', t')]_{z'=0}. \tag{35a}$$

We suppose that the incident field is the pulse (21) with normal incidence so that we are still interested in the total field for $t \geqslant 0$ when $\psi_{\mathrm{i}}(\boldsymbol{u}, t)$ has the expression (22*a*) and, from now on, we denote by $\psi_{\mathrm{r}}^{(0)}(\boldsymbol{u}, t)$ the expression (22*b*) of the pulse reflected according to the Descartes–Snell law.

Our first task is to compute $\phi_{\mathrm{i}}(\boldsymbol{u}, t)$ and we obtain in appendix B

$$\phi_{\mathrm{i}}(\boldsymbol{u}, t) = -2^{-1}(1 + d)[\psi_{\mathrm{i}}(\boldsymbol{u}, t_d) + \psi_{\mathrm{r}}^{(0)}(\boldsymbol{u}, t_d)] \qquad t_d = t - d|z|/c \tag{36}$$

in which $\psi_{\rm i}$ and $\psi_{\rm r}^{(0)}$ are the pulses (22a) and (22b) at the time t_d. The same calculation made with $\psi_{\rm r}^{(0)}$ in the integrand of (35a) would supply $\phi_{\rm r}^{(0)}$ deduced from $\phi_{\rm i}$ by changing z into $-z$ so that

$$\phi_{\rm r}^{(0)}(\boldsymbol{u}, t) = \phi_{\rm i}(\boldsymbol{u}, t). \tag{37}$$

It follows from (36) and (37) that for $d = 0$ we have $\phi_{\rm i} = \phi_{\rm r}^{(0)} = -2^{-1}(\psi_{\rm i} + \psi_{\rm r}^{(0)})$ so that equation (35) reduces to the simple relation $\psi_{\rm i} - 2^{-1}(\psi_{\rm i} + \psi_{\rm r}^{(0)}) = -[\psi_{\rm i} - 2^{-1}(\psi_{\rm i} + \psi_{\rm r}^{(0)})]$.

We now look for the solution of the integral equation (35) with $\psi_{\rm i}$ and $\phi_{\rm i}$ on the right-hand side given respectively by (23a) and (36). A simple approximation of the solution, the first term in fact of a Rayleigh–Gans approximation discussed in [1], is obtained by substituting $\phi_{\rm r}^{(0)}$ for $\phi_{\rm r}$ in (35). Denoting by $\psi_{\rm r}^{(1)}$ this approximation, we obtain

$$\psi_{\rm r}^{(1)}(\boldsymbol{u}, t) = -[\psi_{\rm i}(\boldsymbol{u}, t) + \phi_{\rm i}(\boldsymbol{u}, t) + \phi_{\rm r}^{(0)}(\boldsymbol{u}, t)] \qquad z \leqslant 0 \tag{38}$$

or according to (36) and (37)

$$\psi_{\rm r}^{(1)}(\boldsymbol{u}, t) = (1 + d)[\psi_{\rm i}(\boldsymbol{u}, t_d) + \psi_{\rm r}^{(0)}(\boldsymbol{u}, t_d)] - \psi_{\rm i}(\boldsymbol{u}, t) \qquad z \leqslant 0 \tag{38a}$$

that we write

$$\psi^{(1)}(\boldsymbol{u}, t) = (1 + d)\psi^{(0)}(\boldsymbol{u}, t_d) \qquad z \leqslant 0 \tag{39}$$

in which $\psi^{(0)}$ and $\psi^{(1)}$ are the total field when the reflected field satisfies respectively the Descartes–Snell law and the integral equation (35). So, at this level of approximation, $\psi^{(1)}$ has at time t the structure of $\psi^{(0)}$ at time t_d with an amplitude multiplied by $1 + d$. Therefore, the boundary conditions (32) give rise through the Green function G_Z, that acts as a delay line, to a distorted reflected pulse in such a way that the total field appears as retarded. However, the relation (43) is valid for the first-order approximation (38), and to higher-order approximations distortion could be different. One would proceed similarly for a $\mathcal{D}$-pulse impinging with the incidence $0 \leqslant \theta < \pi/2$, the time t_d being now defined by the relation $t_d = t - d\cos\theta|z|/c$.

It is not claimed that the boundary conditions (32) correspond to a material of practical use. They were chosen to make calculations tractable to illustrate qualitatively the kind of processes that happen on a imperfectly reflecting surface. More realistic boundary conditions are supplied by an impedance plane as discussed in [1] with N now defined as

$$N = d\varepsilon[s_z^2 + (\varepsilon - 1)s^2] \tag{40}$$

but the presence of s_z^2 in (40) instead of s_z in (32) makes the calculation of the inverse Laplace transform required by the integral equation (34) difficult.

3.2. Numerical experiments

Using the same pulse as in section 2.3, calculations are now performed with the Green function (33) discussed in appendix B. The pulse amplitude $\psi(z, t)$ appears in tables 3 and 4 for a pulse duration $\tau = 3$, for $d = 0.05, 0.1, 0.5$ and the same values of z and t as in the previous tables.

The comparison with the results of section 2.3 shows a reduction in the pulse amplitude.

Table 3. $z = 0$, $\tau = 3$.

t	z	$z + 2$	$z + 4$
$d = 0.05$	0.951	17.4×10^{-3}	0
$d = 0.01$	0.904	16.5×10^{-3}	0
$d = 0.50$	0.606	11.6×10^{-3}	0

Table 4. $z = 10$, $\tau = 3$.

t	z	$z+2$	$z+4$
$d = 0.05$	0.475	8.71×10^{-3}	0
$d = 0.01$	0.452	8.27×10^{-3}	0
$d = 0.50$	0.303	5.55×10^{-3}	0

3.3. $\mathcal{D}$-pulse reflection on a time reversal mirror

Time reversal mirrors (TRMs) are used [10, 11] to convert an acoustic wavefield from a source into a wavefield at the source position. The areas of application include medical imagery, lithotripsy and non-destructive testing and the practical realization of TRM is discussed at length in [10], but we are interested here in the theoretical aspect of this problem.

Assuming that (x, z) is the plane of incidence and the total field null on the mirror so that we may use the Fredholm equation (28), we consider a rectangular pulse ψ_i of duration t_0, launched at $t = 0$ by a source located at $x = 0$, $z = z_0$, which impinges normally on a mirror in the $z = 0$ plane. Such a pulse has the form in which U is the unit step function

$$\psi_i(z, t) = U(ct - z_0 + z) - U(ct - ct_0 - z_0 + z) \tag{41}$$

which reduces on the Σ'-plane $z' = 0$ to

$$\psi_i(0, t') = U(ct' - z_0) - U(ct' - ct_0 - z_0). \tag{42}$$

On a conventional mirror $\psi_{re}(0, t') = \psi_i(0, t')$ and the total field is $\psi(0, t') = 2\psi_i(0, t')$, but on a TRM, starring all the quantities pertaining to TRM reflections, one has

$$\psi^*_{re}(0, t') = T\psi_i(0, t') = U(ct' + z_0) - U(ct' - ct_0 + z_0) \tag{43}$$

in which T is the time-inversion operator; note that according to the CPT theorem [12], where P, C are the parity and charge conjugation operators, one has $TU = PU$ since U is real.

So, assuming $ct_0 > z_0$, we may write the total field on the Σ'-plane

$$\psi^*(0, t') = \psi(0, t')/2 + \phi(0, t')/2 \tag{44}$$

in which $\psi(0, t') = 2\psi_i(0, t')$ while since $U(ct' + z_0) = 1$

$$\phi(0, t') = 2[1 - U(ct' - ct_0 + z_0)] \tag{44a}$$

the relation (44) implies that the total field outside the mirror has the form

$$\psi^*(z, t) = \psi(z, t)/2 + \phi(z, t)/2 \tag{45}$$

with $\psi(z, t) = \psi_i(z, t) + \psi_{re}(z, t)$ the total field supplied by the Descartes–Snell law. Therefore, one has only to look for the contribution of $\phi(0, t')$. Now, according to (44*a*) one has with $b^* = ct_0 - z_0$

$$\int_{-\infty}^{\infty} c\,dt' \exp(-sct')\psi(0, t') = 2\int_a^{b*} \exp(-sct')c\,dt'$$
$$= 2s^{-1}\{1 - \exp[-s(ct_0 - z_0)]\}. \tag{46}$$

With (46), the integral (28*a*) becomes

$$F_\phi(ß, s) = 4\pi\delta(ß)s^{-1}\{1 - \exp[-s(ct_0 - z_0)]\} \tag{47}$$

and we obtain from (28) the contribution $\phi(z, t)$ to the total pulse

$$\phi(z, t) = \mathcal{L}^{-1}\{\Phi_-(z, s)\} + \mathcal{L}^{-1}\{\Phi_+(z, s)\} \tag{48}$$

$$\Phi_{\pm}(z,s) = s^{-1}\exp(\pm sz)\{1-\exp[-s(ct_0 - z_0)]\}. \tag{48a}$$

Using $\mathcal{L}^{-1}\{1/s\} = U(t)$ and the well known property of the Laplace transform for $a > 0$ [9]

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \Rightarrow \mathcal{L}^{-1}\{\exp(-as)F(s)\} = f(t-a)U(t-a) \tag{49}$$

we obtain

$$\phi_-(z,t) = \mathcal{L}^{-1}\{\Phi_-(z,s)\} = U(ct-z) - U(ct-ct_0+z_0-z). \tag{50}$$

To obtain $\mathcal{L}^{-1}\{\Phi_+(z,s)\}$, one cannot use (49), which is no longer valid; one has instead [9]

$$\mathcal{L}^{-1}\left\{\exp(as)\left[F(s) - \int_0^a \exp(-s\tau)f(\tau)\,\mathrm{d}\tau\right]\right\} = f(t+a) \qquad a > 0. \tag{51}$$

Note that at the difference of (49) no unit step function is required since t and a are positive and one obtains easily the intuitively evident result

$$\mathcal{L}^{-1}\{\exp(as)\} = \delta(t+a) \qquad a > 0 \tag{52}$$

so that

$$\phi_+(z,t) = \mathcal{L}^{-1}\{\Phi_+(z,s)\} = 1 - U(ct-ct_0+z_0+z). \tag{53}$$

Substituting (50) and (53) into (45) gives the total pulse due to the TRM reflection of a rectangular pulse

$$\psi^*(z,t) = [\psi_{\mathrm{i}}(z,t) + \psi_{\mathrm{re}}(z,t)]/2 + [\phi-(z,t) + \phi_+(z,t)]/2. \tag{54}$$

Since $U(ct+z_0+z) = 1$ one has $\phi_+(z,t) = \mathrm{CP}\,\psi_{\mathrm{re}}(z,t)$: the TRM reflected field is CP invariant.

All these results hold if $ct_0 > z_0$, but according to (43) $\psi^*_{\mathrm{re}}(0,t') = 0$ for $ct_0 < z_0$ because $U(ct'+z_0) = U(ct'-ct_0+z_0) = 1$ so there is no TRM reflection in this case; in particular a TRM does not reflect a Dirac pulse.

A careful analysis of this result to be discussed elsewhere, where the behaviour of different types of pulse is analysed, shows that a TRM-reflected pulse exists in the region $0 \leqslant z \leqslant z_0$ as soon as the incident field reaches the mirror until the return time to the source, which seems to be in agreement with the experimental observations, while beyond this time the contribution is that of a conventional reflected pulse divided by two. When $ct_0 > z_0$ one has similar results except that the contribution of the TRM-reflected pulse starts at $t = 0$. One may understand this result by noting that the time inversion transforms the tail of the acoustic pulse into a precursor, so the larger t_0 is with respect to the return time t_1 to the source, the quicker is the reaction of the mirror.

4. Discussion

Why are we interested in scattering of pulses on planes? There are at least two reasons. First, digital technology, launching signals with finite duration and energy, and blossoming in communications, generates signals, compelled to satisfy causality, that neither propagate nor interact as harmonic wavepackets used today to describe most of the physical processes in classical optics and in radiowave propagation. Furthermore, modern technology could also generate, at least with a good approximation, some of the nondiffractive solutions of the wave equation recently demonstrated, such as focus wave modes [5–8], Bessel [13], Bessel–Gauss [14] and X pulses [15], among which the last two are serious candidates [16, 17] to be created. Already, the present-day laser technology can produce extremely short and intense pulses [18] containing only a few field oscillations, even only one, whose scattering properties differ substantially from those of quasimonochromatic, many-cycle pulses [19].

On the other hand, there now exist new kinds of material, specially chiral, with electric and magnetic properties different from those of dielectrics and of good conducting metals, most often used in mathematical models of diffracting surfaces. So, foreseeing the interactions of these new signals and materials becomes important, which requires us to solve boundary value problems of wave and Maxwell equations. When data are not supplied on simple surfaces, this difficult task is investigated with the help of models based on ansatze [20] suggested by the solution of simpler idealized problems; the present integral equation approach may be used as a tool for obtaining such ansatze.

It is noticed in [1] that the Fredholm integral equation gives a new approach to diffraction of harmonic waves by plane apertures [21]; work is in progress to generalize this approach to scalar pulses as is its extension to electromagnetic fields. The discussion of the validity of the Descartes–Snell law for pulse reflection on an arbitrary surface is postponed to a later work.

Appendix A

Introducing the variables

$$\lambda = k - \omega/c \qquad \mu = ß + \omega \sin\theta/c \qquad s' = x'\mu/\lambda \tag{A.1}$$

the integral (19) becomes

$$\begin{aligned} F(\lambda,\mu) &= (\lambda/\mu c)\iint_{-\infty}^{\infty} \mathrm{d}r'\,\mathrm{d}s'\,(a-\mathrm{i}r')^{-1/2}\exp[-\mathrm{i}\lambda r' - \mathrm{i}s'\lambda\mu^{-1}(\mu-\lambda\sin\theta) \\ &\quad -(\omega\lambda^2/c\mu^2)\cos^2\theta s'^2(a-\mathrm{i}r')^{-1}] \\ &= (\lambda/\mu c)\int_{-\infty}^{\infty}\mathrm{d}r'(a-\mathrm{i}r')^{-1/2}\exp(-\mathrm{i}\lambda r')f(\lambda,\mu) \end{aligned} \tag{A.2}$$

in which the function $f(\lambda,\mu)$ is

$$\begin{aligned} f(\lambda,\mu) &= \int_{-\infty}^{\infty}\mathrm{d}s'\exp(-Q^2s'^2 - Ps') \\ &= \exp(P^2/4Q^2)\int_{-\infty}^{\infty}\mathrm{d}s'\exp[-Q^2(s'+P/2Q^2)^2] \\ &= \pi^{1/2}/Q\exp(P^2/4Q^2) \end{aligned} \tag{A.3}$$

with the functions P and Q defined by the relations

$$P = \mathrm{i}\lambda\mu^{-1}(\mu-\lambda\sin\theta) \qquad Q = (\omega\lambda^2/c\mu^2)\cos^2\theta(a-\mathrm{i}r')^{-1}. \tag{A.4}$$

Taking into account (A.4), the expression (A.3) becomes

$$f(\lambda,\mu) = (\mu/\lambda\cos\theta)[\pi c\omega^{-1}(a-\mathrm{i}r')]^{1/2}\exp[-\mathrm{i}(a-\mathrm{i}r')(\mu-\lambda\sin\theta)(4\omega\cos^2\theta)^{-1}]. \tag{A.5}$$

Introducing the function

$$h(\lambda,\mu) = \lambda - c(\mu-\lambda\sin\theta)^2(4\omega\cos^2\theta)^{-1} \tag{A.6}$$

and substituting (A.5) into (A.2) gives

$$\begin{aligned} F(\lambda,\mu) &= (\pi/cw\cos^2\theta)^{1/2}\exp[-a\{\lambda-h(\lambda,\mu)\}]\int_{-\infty}^{\infty}\mathrm{d}r'\exp[-\mathrm{i}r'h(\lambda,\mu)] \\ &= 2\pi(\pi/cw\cos^2\theta)^{1/2}\exp(-a\lambda)\delta\{h(\lambda,\mu)\}. \end{aligned} \tag{A.7}$$

δ is the Dirac distribution and we used the relation $f(x)\delta(x) = f(0)\delta(x)$. Now the integral (18) becomes in terms of the parameters λ, μ and writing $2\cos(kz) = \exp(ik_z z) + \exp(-ik_z z)$

$$\psi(\boldsymbol{u}, t) = \psi_+(\boldsymbol{u}, t) + \psi_-(\boldsymbol{u}, t) \tag{A.8}$$

$$\psi_\pm(\boldsymbol{u}, t) = (-c/\pi)\exp[i\omega c^{-1}(ct - x\sin\theta)]\Phi_\pm(\boldsymbol{u}, t) \tag{A.8$'$}$$

$$\Phi_\pm(\boldsymbol{u}, t) = \iint_{-\infty}^{\infty} d\lambda\, d\mu \exp[i\lambda ct + i\mu x \pm ik_z z]F(\lambda, \mu) \tag{A.9}$$

with $F(\lambda, \mu)$ given by (A.7) and

$$k_z = [(\lambda + \omega/c)^2 - (\mu - \sin\theta\omega/c)^2]^{1/2}. \tag{A.10}$$

Assuming $\lambda > 0$, we introduce the variable ρ and the parameter b

$$\rho = \mu - \lambda\sin\theta \qquad b = c/4\omega\cos^2\theta \tag{A.11}$$

so that

$$h(\lambda, \mu) = \lambda - b\rho^2 \qquad k_z = [(\lambda + \omega/c)^2 - \{\rho + (\lambda - \omega/c)\sin\theta\}^2]^{1/2}. \tag{A.12}$$

Using (A.7), (A.11) and (A.12), the expression (A.9) of $\Phi_\pm(\boldsymbol{u}, t)$ becomes

$$\begin{aligned}\Phi_\pm(\boldsymbol{u}, t) &= 2\pi(\pi/\omega c\cos^2\theta)^{1/2}\int_{-\infty}^{\infty} d\rho \exp(i\rho x)\int_{-\infty}^{\infty} d\lambda \\ &\quad \times \exp[-a\lambda + i\lambda ct + i\lambda x\sin\theta]\exp(\pm ik_z z)\delta(\lambda - \beta r^2) \\ &= 2\pi(\pi/\omega c\cos^2\theta)^{1/2}\int_{-\infty}^{\infty} d\rho \exp(i\rho x) \\ &\quad \times \exp[-b\rho^2(a - ict - ix\sin\theta)]\exp(\pm ik_z z).\end{aligned} \tag{A.13}$$

Now, according to (A.12) we obtain for k_z in terms of $\lambda = b\rho^2$

$$\begin{aligned}k_z &= [(b\rho^2 - \omega/c)^2\cos^2\theta - 2\rho\sin\theta(b\rho^2 - \omega/c) + \rho^2\tan g^2\theta]^{1/2} \\ &= \cos\theta(b\rho^2 - \omega/c) - \rho\tan g\theta.\end{aligned} \tag{A.14}$$

Introducing the variables

$$e = i(x - z\tan g\theta) \qquad g = a - ict - ix\sin\theta - iz\cos\theta \tag{A.15}$$

and substituting (A.14) into (A.13) gives

$$\begin{aligned}\Phi_+(\boldsymbol{u}, t) &= 2\pi(\pi/\omega c\cos^2\theta)^{1/2}\exp(-i\omega c^{-1}\cos\theta z)\int_{-\infty}^{\infty} d\rho \exp(-bg\rho^2 + e\rho) \\ &= 2\pi(\pi/wc\cos\theta)^{1/2}\exp(-i\omega c^{-1}\cos\theta z)[(\pi/bg)^{1/2}\exp(e^2/4bg)].\end{aligned} \tag{A.16}$$

Then, using the expressions (A.11) for b and (A.15) for e, we obtain

$$\Phi_+(\boldsymbol{u}, t) = 4\pi^2 c^{-1}g^{-1/2}\exp(-i\omega c^{-1}z\cos\theta)\exp[-\omega c^{-1}g^{-1}(x\cos\theta - z\sin\theta)^2]. \tag{A.17}$$

One has just to change z into $-z$ in (A.17) to obtain Φ_-

$$\Phi_-(\boldsymbol{u}, t) = 4\pi^2 c^{-1}g^{-1/2}\exp(i\omega c^{-1}z\cos\theta)\exp[-\omega c^{-1}g^{-1}(x\cos\theta + z\sin\theta)^2]. \tag{A.18}$$

Substituting (A.17) into (A.8$'$) gives

$$\psi_+(\boldsymbol{u}, t) = -g^{-1/2}\exp[i\omega c^{-1}(ct - x\sin\theta - z\cos\theta) - \omega c^{-1}g^{-1}(x\cos\theta - z\sin\theta)^2] \tag{A.19}$$

that is, $\psi_+ = \psi_i$ and similarly $\psi_- = \psi_r$ so that according to (A.8) $\psi = \psi_i + \psi_r$.

Appendix B

We obtain from (23) and (33) with the expression (23a) of $\Phi\{ß, s)$

$$8\mathrm{i}\pi^2 G_Z(\boldsymbol{u}, t; \boldsymbol{u}', t') = c\int_{\mathrm{Br}} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}ß\, s_z^{-1}\Phi(ß, s)\{\exp(-s_z|z-z'|) + \exp(-s_z|z+z'|)\}\exp(-s_z d||z-z'|) \tag{B.1}$$

so that according to (24) and (25)

$$4\mathrm{i}\pi^2[\partial_{z'} G_Z(\boldsymbol{u}, t; \boldsymbol{u}', t')]_{z'=0} = -c(1+d)\int_{\mathrm{Br}} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}ß\, \Phi(ß, s)\cosh(s_z z)\exp(-s_z d|z|) \tag{B.2}$$

while according to (22a)

$$[\psi_{\mathrm{i}}(\boldsymbol{u}', t')]_{z'=0} = f(t')[U(t') - U(t'-z')]. \tag{B.3}$$

Substituting (B.2) and (B.3) into (35a) gives

$$4\mathrm{i}\pi^2\phi_{\mathrm{i}}(\boldsymbol{u}, t) = -c(1+d)\int_{\mathrm{Br}} \mathrm{d}s\, \exp(sct) \times \int_{-\infty}^{\infty} \mathrm{d}ß\, F(ß, s)\exp(\mathrm{i}ßx)\cosh(s_z z)\exp(-s_z \mathrm{d}|z|) \tag{B.4}$$

with $F(ß, s)$ given by (28b) so that

$$\begin{aligned} 4\mathrm{i}\pi\phi_{\mathrm{i}}(\boldsymbol{u}, t) &= -2c(1+d)\int_0^{\tau} \mathrm{d}t' f(t')\int_{\mathrm{Br}} \mathrm{d}s\, \exp(sct - sct') \\ &\quad \times \int_{-\infty}^{\infty} \mathrm{d}ß\, \delta(ß)\exp(\mathrm{i}ßx)\cosh(s_z z)\exp(-s_z d|z|) \\ &= -c(1+d)\int_0^{t} \mathrm{d}t' f(t')\int_{\mathrm{Br}} \mathrm{d}s\, \exp[sc(t - d|z|/c) - sct'] \\ &\quad \times\{\exp(sz) + \exp(-sz)\}. \end{aligned} \tag{B.5}$$

Taking into account (29a), (30) and (31), the comparison of (B.5) and (29) shows that

$$2\phi_{\mathrm{i}}(\boldsymbol{u}, t) = -(1+d)[\psi_{\mathrm{i}}(\boldsymbol{u}, t_d) + \psi_{\mathrm{r}}^{(0)}(\boldsymbol{u}, t_d)] \qquad t_d = t - d|z|/c \tag{B.6}$$

in which ψ_{i} and $\psi_{\mathrm{r}}^{(0)}$ are the fields (22a) and (22b) respectively.

Acknowledgment

I am indebted to Dr G Nurdin for many discussions on the numerical inversion of the Fourier and Laplace transforms.

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